Electric and Electronic Engineering

4. AC Network Analysis

Sinusoidal (or AC) signals constitute the most important class of signals in the analysis of electrical circuits. The simplest reason is that virtually all of the electric power used in households and industries comes in the form of sinusoidal voltages and currents.

4.1. Impedance of a Practical Capacitor

A practical capacitor can be modeled by an ideal capacitor in parallel with a resistor (Fig.38).

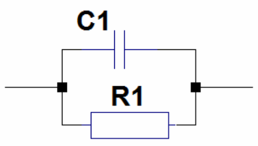


Fig. 38 – Equivalent circuit of the practical capacitor

The parallel resistance represents leakage losses in the capacitor and is usually quite large. Let find the impedance of a practical capacitor at the radian frequency $\omega = 314$ rad/s. How will the impedance change if the capacitor is used at a much higher frequency, say 8 MHz? Consider the real values $C_1 = 0.1\ \mu\text{F} = 0.1 \times 10^{-6}$ F; $R_1 = 1$ MOhm.

To determine the equivalent impedance we combine the two impedances in parallel:

$$Z_1 = R_1 \left\| \frac{1}{j\omega C_1} \right. = \frac{R_1 \dfrac{1}{j\omega C_1}}{R_1 + \dfrac{1}{j\omega C_1}} = \frac{R_1}{1 + j\omega C_1 R_1}$$

Substituting numerical values, we find

$Z_1(\omega=314) = 10^6/(1+j314\cdot10^6\cdot0.1\cdot10^{-6}) = 10^6/(1+j31.4)$.

The impedance of the capacitor alone at this frequency would be:

$Z_{C1}(\omega = 314) = 1/(j314 \cdot 0.1 \cdot 10^{-6}) = 26.53 \cdot 10^3\angle{-\pi/2}$ Ohm

If the frequency is increased to 8 MHz, or $16\pi \times 10^6$ rad/s—a radio frequency—we can recompute the impedance to be:

$Z_1(\omega = 16\pi \cdot 10^6) = 10^6 /(1 + j16\pi \cdot 10^6 \cdot 0.1 \cdot 10^{-6} \times 10^6) =$

$10^6/(1 + j160\pi\cdot 10^6) = 0.2\angle{-1.57}$ Ohm

The impedance of the capacitor alone at this frequency would be:

$Z_{C1}(\omega = 16\pi \cdot 10^6) = 1/(j16\pi\cdot 10^6 \cdot 0.1\cdot 10^{-6}) = 0.2 \angle{-\pi/2}$ Ohm

Note that the effect of the parallel resistance at the lower frequency (corresponding to the well-known 50-Hz AC power frequency) is significant: The effective impedance of the practical capacitor is substantially different from that of the ideal capacitor. On the other hand, at much higher frequency, the parallel resistance has an impedance so much larger than that of the capacitor that it effectively acts as an open circuit, and there is no difference between the ideal and practical capacitor impedances.

This example suggests that the behavior of a circuit element depends very much in the frequency of the voltages and currents in the circuit. We should also note that the inductance of the wires may become significant at high frequencies.

4.2. Impedance of a Practical Inductor

A practical inductor can be modeled by an ideal inductor in series with a resistor. The series resistance represents the resistance of the coil wire and is usually small. Let us find the range of frequencies over which the impedance of this practical inductor is largely *inductive* (i.e., due to the inductance in the circuit). We shall consider the impedance to be inductive if the impedance of the inductor in the circuit of Figure 39 is at least 10 times as large as that of the resistor.

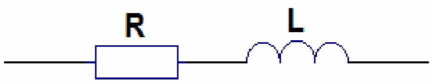


Fig. 39– Equivalent network of the practical inductor

The real values are L = 0.098 H; lead length = lc = 2 × 10 cm; n = 250 turns; the wire diameter is 0.2 mm; the coil intersection is 0.25 × 0.5 cm . Resistance of such a wire is = 0.558 Ohm/m.

Analysis: We first determine the equivalent resistance of the wire used in the practical inductor using the cross section as an indication of the wire length, lw, used in the coil:

lw = 250 × (2 × 0.25 + 2 × 0.5) = 375 cm

Total length l = = $lw + lc$ = 375 + 20 = 395 cm

The total resistance is therefore

R = 0.558 Ohm/m × 0.395 m = 0.22 Ohm.

Thus, we wish to determine the range of radian frequencies, ω, over which the magnitude of $j\omega L$ is greater than 10 × 0.22 Ohm:

$\omega L > 2.2$, or $\omega > 2.2/L = 2.2/0.098 = 22.4$ rad/s.

Alternatively, the range is $f = \omega/2\pi = 3.56$ Hz.

Note how the resistance of the coil wire is relatively insignificant. This is true because the inductor is rather large; wire resistance can become significant for very small inductance values. At high frequencies, a capacitance should be added to the model because of the effect of the insulator separating the coil wires.

4.3. Impedance of a More Complex Circuit

Consider the following problem. Find the equivalent impedance of the circuit shown in Figure 40 by the following parameters: ω = 104 rad/s; $R1$ = 100 Ohm; L = 10 mH; $R2$ = 50 Ohm, C = 10 μF.

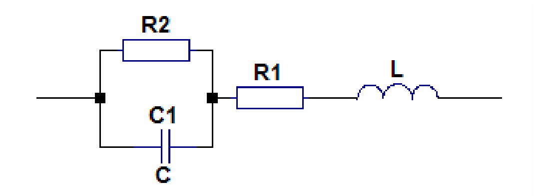


Fig. 40 – Example of the circuit to analyze

Analysis: We determine first the parallel impedance of the R_2-C circuit, $Z_{||}$.

$$Z_{||} = R_2 \left\| \frac{1}{j\omega C} \right. = \frac{R_2 \dfrac{1}{j\omega C}}{R_2 + \dfrac{1}{j\omega C}} = \frac{R_2}{1 + j\omega C R_2} = 50/(1 + j10^4 \cdot 10 \cdot 10^{-6} \cdot 50) = 50/(1 + j5) =$$

$$= 1.92 - j9.62 = 9.81\angle{-1.3734} \text{ Ohm}.$$

Next, we determine the equivalent impedance, Z_{eq}:

$Z_{eq} = R_1 + j\omega L + Z_{||} = 100 + j104 \times 10^{-2} + 1.92 - j9.62$

$= 101.92 + j90.38 = 136.2\angle 0.723$ -

Is this impedance inductive or capacitive in nature? At the frequency used in this example, the circuit has an inductive impedance, since the reactance is positive (or, alternatively, the phase angle is positive).

4.4. Admittance

In Chapter 1.3, it was suggested that the solution of certain circuit analysis problems was handled more easily in terms of conductances than resistances. In AC circuit analysis, an analogous quantity may be defined, the reciprocal of complex impedance. Just as the conductance, G, of a resistive element was defined as the inverse of the resistance, the **admittance** of a branch is defined as follows:

$$Y = 1/Z.$$

Note immediately that whenever Z is purely real—that is, when $Z = R+j0$—the admittance Y is identical to the conductance G. In general, however, Y is the complex number

$$Y = G + jB$$

where G is called the **AC conductance** and B is called the **susceptance;** the latter plays a role analogous to that of reactance in the definition of impedance. Clearly, G and B are related to R and X. However, this relationship is not as simple as an inverse. Let $Z = R + jX$ be an arbitrary impedance. Then, the corresponding admittance is

$$Y = \frac{1}{Z} = \frac{1}{R + jX}$$

In order to express Y in the form $Y = G + jB$, we multiply numerator and denominator by $R-jX$:

$$Y = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

and conclude that

$$G = \frac{R}{R^2 + X^2}; \qquad B = \frac{-X}{R^2 + X^2}.$$

Notice in particular that G *is not the reciprocal of* R in the general case!

4.5. AC circuit analysis

This section will illustrate how the use of phasors and impedance facilitates the solution of AC circuits by making it possible to use the same solution methods developed in Chapter 3 for DC circuits. The AC circuit analysis problem of interest in this section consists of determining the unknown voltage (or currents) in a circuit containing linear passive circuit elements (R, L, C) and excited by a sinusoidal source. Figure 41 depicts one such circuit, represented in both conventional time-domain form and phasor-impedance form.

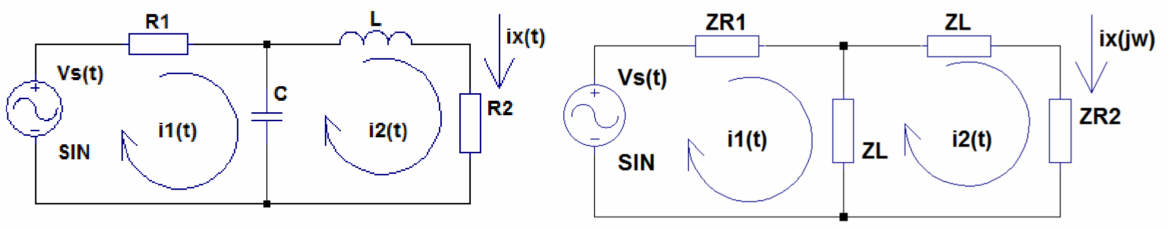


Fig. 41 – Circuit, represented in both conventional form and phasor-impedance form

The first step in the analysis of an AC circuit is to note the frequency of the sinusoidal excitation. Next, all sources are converted to phasor form, and each circuit element to impedance form. This is illustrated in the phasor circuit of Figure 41. At this point, if the excitation frequency, ω, is known numerically, it will be possible to express each impedance in terms of a known amplitude and phase, and a numerical answer to the problem will be found. It does often happen, however, that one is interested in a more general circuit solution, valid for an arbitrary excitation frequency. In this latter case, the solution becomes a function of ω. This point will be developed further in Chapter 5, where the concept of sinusoidal frequency response is discussed.

With the problem formulated in phasor notation, the resulting solution will be in phasor form and will need to be converted to time-domain form. In effect, the use of phasor notation is but an intermediate step that greatly facilitates the computation of the final answer. In summary, here is the procedure that will be followed to solve an AC circuit analysis problem.

Method of AC Circuit Analysis

1. Identify the sinusoidal source(s) and note the excitation frequency.
2. Convert the source(s) to phasor form.
3. Represent each circuit element by its impedance.
4. Solve the resulting phasor circuit, using appropriate network analysis tools.
5. Convert the (phasor-form) answer to its time-domain equivalent, using equation 4.46.

4.6. Example of the AC analysis

Apply the phasor analysis method just described to the circuit of Figure 42 to determine the source current. The parameters are ω = 100 rad/s; $R1$ = 50 Ohm; $R2$ = 200 Ohm, C = 100 μF. The source current $i_S(t)$ has to be found.

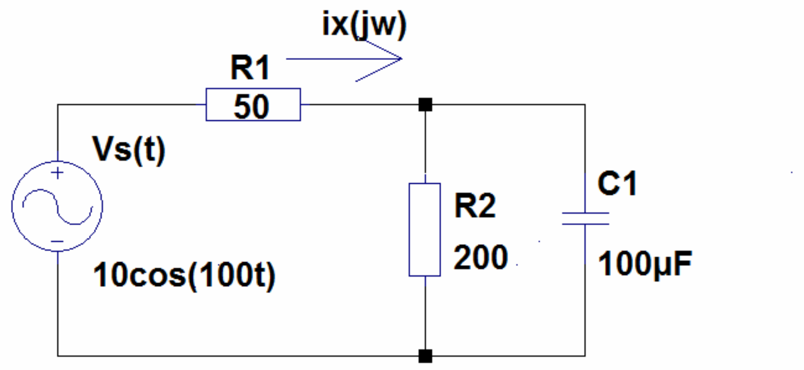


Fig. 42 – Circuit to analyze

Analysis: Define the voltage *v* at the top node and use nodal analysis to determine *v*. Then observe that

$$i_S(t) = \frac{v_S(t) - v(t)}{R_1}$$

Next, we follow the steps outlined above.

Step 1: *vS (t)* = 10 cos(100*t)* V; $\omega = 100$ rad/s.

Step 2: $v_S(j\omega) = 10\angle 0$ V.

Step 3: $Z_{R1} = 50$ Ohm, $Z_{R2} = 200$ Ohm, $Z_C = 1/(j100 \times 10^{-4}) = -j100$ Ohm. The resulting phasor circuit is shown in Figure 43.

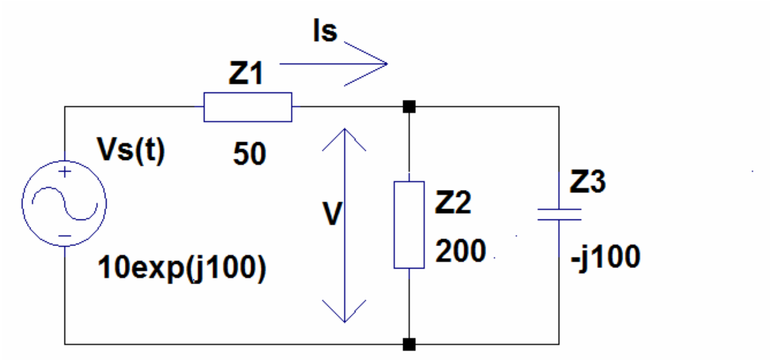


Fig. 43 – Phasor equivalent of the circuit in Fig. 42.

Step 4: Next, we solve for the source current using nodal analysis. First we find **V**:

$$\boldsymbol{I}_S = \frac{\boldsymbol{V}_S - \boldsymbol{V}}{Z_1} = \frac{\boldsymbol{V}}{Z_2 \parallel Z_3};$$

$$\frac{\boldsymbol{V}_S}{Z_1} = \boldsymbol{V}\left(\frac{1}{Z_2 \parallel Z_3} + \frac{1}{Z_1}\right);$$

$$\boldsymbol{V} = \left(\frac{1}{Z_2 \parallel Z_3} + \frac{1}{Z_1}\right)^{-1} \frac{\boldsymbol{V}_S}{Z_1} = \left(\frac{1}{40 - j80} + \frac{1}{50}\right)^{-1} \frac{\boldsymbol{V}_S}{50} = 7.43\angle -0.381\,\text{V}.$$

Then we compute $\boldsymbol{I}_S$:

$$\mathbf{I}_S = \frac{\mathbf{V}_S - \mathbf{V}}{Z_1} = 0.083\angle 0.727\ \text{A}.$$

Step 5: Finally, we convert the phasor answer to time domain notation:

$i_s(t) = 0.083\cos(100t + 0.727)$ A.

By now it should be apparent that the laws of network analysis introduced in Chapter 3 are also applicable to phasor voltages and currents. This fact suggests that it may be possible to extend the node and mesh analysis methods developed earlier to circuits containing phasor sources and impedances, although the resulting simultaneous complex equations are difficult to solve without the aid of a computer, even for relatively simple circuits.

On the other hand, it is very useful to extend the concept of equivalent circuits to the AC case, and to define complex Th´evenin (or Norton) equivalent impedances. The fundamental difference between resistive and AC equivalent circuits is that the AC Th´evenin (or Norton) equivalent circuits will be frequency-dependent and complex-valued. In general, then, one may think of the resistive circuit analysis of Chapter 3 as a special case of AC analysis in which all impedances are real.

4.7. AC Equivalent Circuits

In Chapter 3, we demonstrated that it was convenient to compute equivalent circuits, especially in solving for load-related variables. Figure 44 depicts the two representations analogous to those developed in Chapter 3. Figure 44(a) shows an *equivalent load*, as viewed by the source, while Figure 44(b) shows an *equivalent source* circuit, from the perspective of the load.

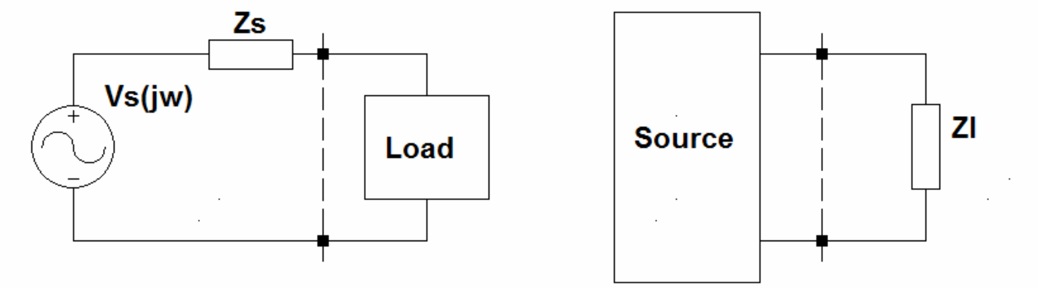
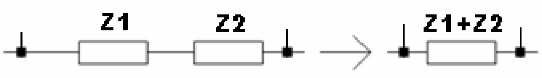


Fig. 44 – Equivalent load (a), and equivalent source (b)

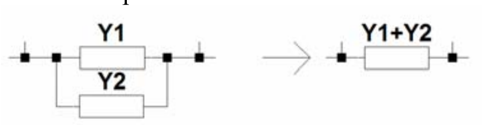
In the case of linear resistive circuits, the equivalent load circuit can always be expressed by a single equivalent resistor, while the equivalent source circuit may take the form of a Norton or a Th´evenin equivalent. This section extends these concepts to AC circuits and demonstrates that the notion of equivalent circuits applies to phasor sources and impedances as well. The techniques described in this section are all analogous to those used for resistive circuits, with resistances replaced by impedances, and arbitrary sources replaced by phasor sources. The principal difference between resistive and AC equivalent circuits will be that the latter are frequency-dependent. Figure 45 summarizes the fundamental principles used in computing an AC equivalent circuit.

Note the definite analogy between impedance and resistance elements, and between conductance and admittance elements.

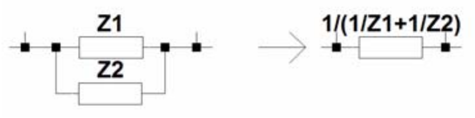
Impedances in series add



Adnittances in parallel add



Impedances in parallel behave like resistors in parallel:



Admittances in series behave like conductances in series

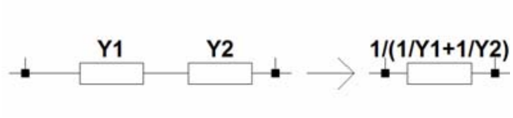


Fig. 45 – Serial and parallel connections of the impedances and admittances

The computation of an equivalent impedance is carried out in the same way as that of equivalent resistance in the case of resistive circuits:

1. Short-circuit all voltage sources, and open-circuit all current sources.
2. Compute the equivalent impedance between load terminals, with the load disconnected.

In order to compute the Th´evenin or Norton equivalent form, we recognize that the Th´evenin equivalent voltage source is the open-circuit voltage at the load terminals and the Norton equivalent current source is the short-circuit current (the current with the load replaced by a short circuit).

The remainder of the section will consist of an example aimed at clarifying some of the finer points in the calculation of such equivalent circuits. Note how the initial circuit reduction proceeds exactly as in the case of a resistive circuit; the details of the complex algebra required in the calculations are explored in the examples.

4.8.Solution of AC Circuit by Nodal Analysis

The electrical characteristics of electric motors can be approximately represented by means of a series R-L circuit. In this problem we analyze the currents drawn by two different motors connected to the same AC voltage supply (Figure 46).

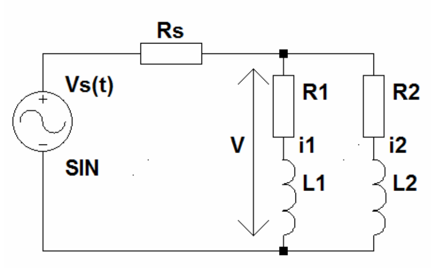


Fig. 46 – Equivalent circuit of two motors connection

The parameters are: R_S = 0.5 Ohm; R_1 = 2 Ohm; R_2 = 0.2 Ohm, L_1 = 0.1 H; L_2 = 20 mH. $v_S(t)$ = 310 cos(314*t*) V. The motor load currents, $i_1(t)$ and $i_2(t)$ have to be found.

Analysis: First, we calculate the impedances of the source and of each motor:

ZS = 0.5 Ohm

$Z1 = 2 + j314 \cdot 0.1 = 2 + j31.4 = 31.46\angle 1.52$ Ohm

$Z2 = 0.2 + j314 \cdot 0.02 = 0.2 + j6.28 = 6.28\angle 1.54$ Ohm

The source voltage is $\boldsymbol{V}_S = 310\angle 0$ V.

Next, we apply Kirchhoff's current law at the top node, with the aim of solving for the node voltage $\boldsymbol{V}$:

$$(\boldsymbol{V}_S - \boldsymbol{V})/Z_S = \boldsymbol{V}/Z_1 + \boldsymbol{V}/Z_2;$$

$$\boldsymbol{V}_S/Z_S = \boldsymbol{V}/Z_S + \boldsymbol{V}/Z_1 + \boldsymbol{V}/Z_2 = \boldsymbol{V}\,(1/Z_S + 1/Z_1 + 1/Z_2);$$

$$\boldsymbol{V} = (1/Z_S + 1/Z_1 + 1/Z_2)^{-1} \cdot \boldsymbol{V}_S/Z_S = \boldsymbol{V}_S\,(1/0.5 + 1/(2+j37.7) + 1/(0.2 + j7.54))^{-1}/0.5 =$$

$$= 308.2\angle 0.079 \text{ V};$$

Having computed the phasor node voltage, **V**, we can now easily determine the phasor motor currents, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$:

$\boldsymbol{I}_1 = \boldsymbol{V}/Z_1 = (82\angle{-0.305})/(2 + j37.7) = 4.08\angle{-1.439}$;

$\boldsymbol{I}_2 = \boldsymbol{V}/Z_2 = (82\angle{-0.305})/(2 + j7.54) = 20.44\angle{-1.465}$.

Finally, we can write the time-domain expressions for the currents:

$i1(t) = 8.166 \cos(314t - 1.439) = 8.166 \cos(314t - 82°)$A

$i2(t) = 40.88 \cos(314t - 1.465) = 40.88 \cos(314t - 84°)$A

Note the phase shift between the source voltage and the two motor currents.

In this chapter we have introduced concepts and tools useful in the analysis of AC circuits. The importance of AC circuit analysis cannot be overemphasized, for a number of reasons.

First, circuits made up of resistors, inductors, and capacitors constitute reasonable models for more complex devices, such as transformers, electric motors, and electronic amplifiers.

Second, sinusoidal signals are ever present in the analysis of many physical systems, not just circuits.

5 Quadripoles

5.1. Quadripole parameters

The part of the electrical circuit, which has two couples of contacts or terminals is named the four pole circuit or shortly **quadripole**. Often electric filters, amplifiers, long lines, transformers are considered as the quadripoles. The contacts, or terminals where the energy source is attached, are named as **input** ones. And the contacts where the loading is attached are named as **output** ones. In shorts, they usually named as input and output of the quadripoles.

If all the elements of the quadripole are linear ones then such a circuit is named as the **linear** quadripole. If the elements of the quadripole are nonlinear ones, then it is named as **nonlinear** one.

The quadripole is named as **active** one if it contains the electric energy source, and its presence can be derived by some measurements in its terminals. Another circuits are named as **passive** ones.

Equivalent quadripoles can substitute each other in the electric circuit without exchanging the currents and voltages in the rest of that circuit.

Complex electrical circuit, which has inputs and outputs can be considered as a set of quadripoles which are coupled according to some structure. The quadripole theory helps to calculate the properties of such a complex circuit using the parameters of its composing subcircuits. This theory supports the solving the circuit synthesis problem, i.e. to find out a set of quadripoles and the structure of its connections [1].

Consider the quadripole which does not contains the independent energy sources. Its left contact couple is signed as 1-1', and the right couple of contacts is signed as 2-2'. Such a circuit is drawn on the fig. 47.

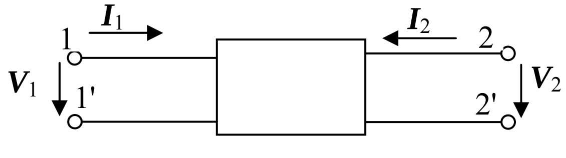


Fig. 47. Quadripole

The dependences between voltages and currents in the input and output of the quadripole can be shown by the following forms of linear equations.

The $||Z||$ form: $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are expressed through $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$:

$$\left.\begin{aligned} \boldsymbol{V}_1 &= Z_{11}\,\boldsymbol{I}_1 + Z_{12}\boldsymbol{I}_2; \\ \boldsymbol{V}_2 &= Z_{21}\,\boldsymbol{I}_1 + Z_{22}\boldsymbol{I}_2. \end{aligned}\right\} \qquad (6)$$

Coefficients Z have the units as the resistance has. They are named usually as impedance. On the contrast to the resistance, the impedance has the real (resistance) and imaginary (reactance) parts.

Coefficients Z can be derived as follows.

$Z_{11}= (\boldsymbol{V}_1/\boldsymbol{I}_1)|_{I2=0}$ is the input impedance in contacts 1 when contacts 2 are open.

$Z_{22}= (\boldsymbol{V}_2/\boldsymbol{I}_2)|_{I1=0}$ is the input resistance in contacts 2 when contacts 1 are open.

$Z_{12}= (\boldsymbol{V}_1/\boldsymbol{I}_2)\,|_{I1=0}$ is the mutual resistance when contacts 1 are open.

$Y_{21}= (\boldsymbol{V}_2/\boldsymbol{I}_1)|_{I2=0}$ is the mutual resistance when contacts 2 are open.

Form $||Y||$: $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are expressed in dependence on $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$:

$$\left.\begin{aligned} \boldsymbol{I}_1 &= Y_{11}\boldsymbol{V}_1 + Y_{12}\boldsymbol{V}_2; \\ \boldsymbol{I}_2 &= Y_{21}\boldsymbol{V}_1 + Y_{22}\boldsymbol{V}_2. \end{aligned}\right\} \qquad (7)$$

Coefficients Y represent the input **admittance** and mutual admittance of input and output. The admittance is the complex value which consists of real (conductance) and imaginary (susceptance) parts.

$Y_{11}= (\boldsymbol{I}_1/\boldsymbol{V}_1)|_{V2=0}$ is the input admittance in contacts 1 when contacts 2 are shorted out.

$Y_{22}= (\boldsymbol{I}_2/\boldsymbol{V}_2)|_{V1=0}$ is the input admittance in contacts 2 when contacts 1 are shorted out.

$Y_{12}= (\boldsymbol{I}_1/\boldsymbol{V}_2)|_{V1=0}$ is the mutual admittance when contacts 1 are shorted out.

$Y_{21}= (\boldsymbol{I}_2/\boldsymbol{V}_1)|_{V2=0}$ is the mutual admittance when contacts 2 are shorted out.

If the quadripole is **reversible** one then $Y_{12}= Y_{21}$. In the reversible quadripole its input and output can exchange each other. If the quadripole is **symmetric** one then $Y_{12}= Y_{21}$ and $Y_{11}= Y_{22}$.

The $||H||$ form: $\boldsymbol{V}_1$ and $\boldsymbol{I}_2$ are expressed through $\boldsymbol{I}_1$ and $\boldsymbol{V}_2$:

$$\left.\begin{aligned} \boldsymbol{V}_1 &= H_{11}\boldsymbol{I}_1 + H_{12}\boldsymbol{V}_2; \\ \boldsymbol{I}_2 &= H_{21}\boldsymbol{I}_1 + H_{22}\boldsymbol{V}_2. \end{aligned}\right\} \qquad (8)$$

Coefficients H can be derived as follows.

$H_{11}= (\boldsymbol{V}_1/\boldsymbol{I}_1)|\boldsymbol{V}_2=0$ is the input resistance when the contacts 2 are shortened on.

$H_{12}= (\boldsymbol{V}_1/\boldsymbol{V}_2)|\boldsymbol{I}_1=0$ is the **feedback factor** when contacts 1 are open.

$H_{21}= (\boldsymbol{I}_2/\boldsymbol{I}_1)|\,\boldsymbol{V}_2=0$ is the **gain factor** when contacts 2 are shorten on.

$H_{22}= (\boldsymbol{V}_2/\boldsymbol{I}_2)|\boldsymbol{I}_1=0$ is the output conductivity when contacts 1 are open.

The $||A||$ form: $\boldsymbol{V}_1$ and $\boldsymbol{I}_1$ are expressed through $\boldsymbol{I}_2$ and $\boldsymbol{V}_2$:

$$\left.\begin{aligned} \boldsymbol{V}_1 &= A_{11}\boldsymbol{V}_2 - A_{12}\boldsymbol{I}_2; \\ \boldsymbol{I}_1 &= A_{21}\boldsymbol{V}_2 - A_{22}\boldsymbol{I}_2. \end{aligned}\right\} \qquad (9)$$

If the quadripole is **reversible** one then the discriminant is $|A| = A_{11}A_{22} - A_{12}A_{21} = 1$. If it is **symmetric** one then $A_{11} = A_{22}$.

Some substitution circuits can be built, which are based on the quadripole equations. The Π – imaged circuit is shown on the fig. 48.

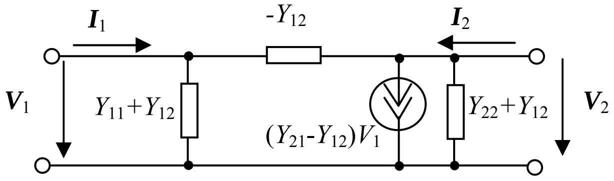


Fig. 48. The Π – imaged circuit of the quadripole

Note that the dependent current source $(Y_{21} - Y_{12})\boldsymbol{V}_1$ is present if the quadripole is unreversible one. The edge admitances are derived from the Y – coefficients.

The T –imaged circuit is shown on the fig. 49:

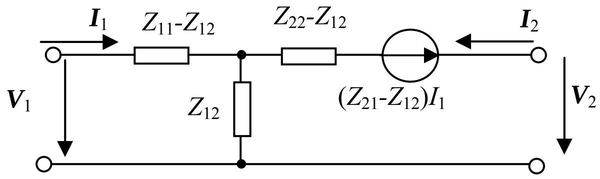


Fig. 49. The Π – imaged circuit of the quadripole

Here the dependent voltage source $(Z_{21} - Z_{12})\boldsymbol{I}_1$ is present if the quadripole is unreversible one. The edge impedances are derived from the Z – coefficients.

One parameter set can be derived from another parameter set. These parameter sets can be derived as the following:

$$\begin{aligned}
H_{11} &= |Z|/Z_{22} = 1/Y_{11} = A_{12}/A_{22}; & Z_{11} &= A_{11}/A_{21} = Y_{22}/|Y| = |H|/H_{22}; \\
H_{12} &= Z_{12}/Z_{22} = -Y_{12}/Y_{11} = |A|/A_{22}; & Z_{12} &= |A|/A_{21} = -Y_{12}/|Y| = H_{12}/H_{22}; \\
H_{21} &= -Z_{21}/Z_{22} = Y_{21}/Y_{11} = -1/A_{22}; & Z_{21} &= 1/A_{21} = -Y_{21}/|Y| = -H_{21}/H_{22}; \\
H_{22} &= 1/Z_{22} = |Y|/Y_{11} = A_{21}/A_{22}; & Z_{22} &= A_{22}/A_{21} = Y_{11}/|Y| = 1/H_{22}; \qquad (10) \\
A_{11} &= Z_{11}/Z_{21} = -Y_{22}/Y_{21} = |H|/H_{21}; & Y_{11} &= A_{22}/A_{12} = Z_{22}/|Z| = 1/H_{11}; \\
A_{12} &= |Z|/Z_{21} = -1/Y_{21} = -H_{11}/H_{21}; & Y_{12} &= -|A|/A_{12} = -Z_{12}/|Z| = -H_{12}/H_{11}; \\
A_{21} &= 1/Z_{21} = |Y|/Y_{21} = -H_{22}/H_{21}; & Y_{21} &= -1/A_{12} = -Z_{21}/|Z| = H_{21}/H_{11}; \\
A_{22} &= Z_{22}/Z_{21} = -Y_{11}/Y_{21} = -1/H_{21}; & Y_{22} &= A_{11}/A_{12} = Z_{11}/|Z| = |H|/H_{11};
\end{aligned}$$

where $|Z| = Z_{11}Z_{22} - Z_{11}Z_{22}$ is the determinant of the matrix of Z – coefficients.

5.2.The input impedance of the loaded quadripole.

The input impedance when the loading is Z_2 (see the fig. 50) is equal to:

$$Z_{1in}=(A_{11}Z_2+A_{12})/(A_{21}Z_2+A_{22}) \tag{11}$$

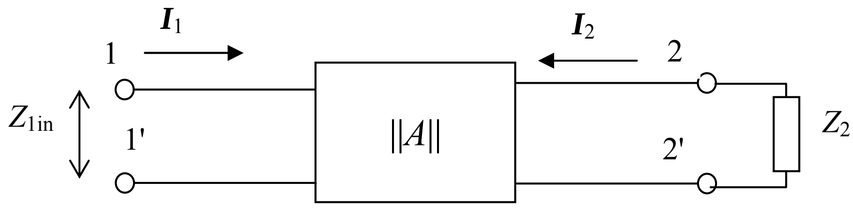


Fig. 50. The quadripole which is loaded by Z_2 impedance

5.3. Characteristic impedance

Consider the quadripole is loaded in the output by the impedance Z_{2C}, and the measured input impedance is Z_{1C}. And on the contrary, this quadripole is loaded in the input by the impedance Z_{1C}, and the measured output impedance is Z_{2C}. (see the fig. above)

Then the values Z_{1C}, Z_{2C} are named as **characteristic impedances** or **surge impedances** of the quadripole. And the condition when the loading has the proper characteristic impedance is named as the loading balance condition. The substitution of these values in (7) and solving the derived equations gives:

$$Z_{1C}=\sqrt{\frac{A_{11}A_{12}}{A_{21}A_{22}}};\, Z_{2C}=\sqrt{\frac{A_{22}A_{12}}{A_{21}A_{11}}}.$$

If the quadripole is symmetric one, then

$$Z_{1C}=Z_{2C}=Z_C=\sqrt{\frac{A_{12}}{A_{21}}}.$$

That means, that if the symmetric quadripole is loaded by the characteristic impedance then the ratios of voltages and currents on its imputs and outputs are equal to this impedance:

$$\boldsymbol{V}_1/\boldsymbol{I}_1=\boldsymbol{V}_2/\boldsymbol{I}_2=Z_C.$$

Consider the energy power at the input $P_1=\boldsymbol{V}_1\boldsymbol{I}_1$ and the output $P_2=\boldsymbol{V}_2\boldsymbol{I}_2$ of the quadripole, then the logarithm of the ratio of these powers is equal to the **attenuation** of the quadripole

$$a=10\lg(P_1/P_2),$$

which is measured in **decibels,** shortly, db. For example, when the quadripole has attenuation a=6,02 db then it attenuates the power in 4 times, and for a=20 db the power is attenuated in 100 times.

When the quadripole is symmetric one and is loaded by the characteristic impedance then

$$P_1/P_2 = (\mathbf{V}_1/\mathbf{V}_2)^2 = (\mathbf{I}_1/\mathbf{I}_2)^2, \text{ and } a = 20 \lg(\mathbf{V}_1/\mathbf{V}_2).$$

For such quadripoles the attenuation a=20 db means that the output voltage is less than the input voltage in 10 times.

Here a is named the **eigen attenuation.** Consider the quadripole is switshed on the circuit (see the fig.51) which is feeded by some voltage source with the impedance Z_S, and is loaded by the impedance Z_L.

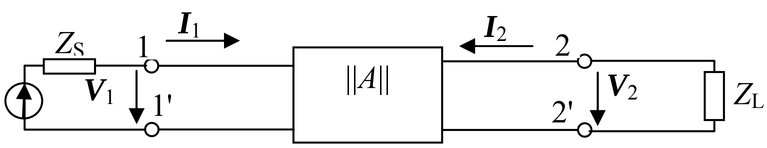


Fig. 51. The quadripole which is connected to some circuit

Then the logarithm of the ratio of powers at the input and output is equal to the **introduced attenuation** of the quadripole

$$a_i = 10 \lg(P_1/P_2) = a + a_1 + a_2 + a_3 + a_4,$$

where a is eigen attenuation, $a_1 = 20 \lg|(Z_S + Z_{1C})/(2\sqrt{(Z_S Z_{1C})})|$ is the attenuation due to the unreconsiled input impedances, $a_2 = 20 \lg|(Z_L + Z_{2C})/(2\sqrt{(Z_L Z_{2C})})|$ is the attenuation due to the unreconsiled output impedances, a_3 is the attenuation due to the mutual unreconsiled impedances, and $a_4 = 20\lg|(Z_S+Z_L)/2\sqrt{(Z_S Z_L)})|$ is the attenuation due to the unreconsiled input and output impedances.

As we can see, the attenuation is minimized when all the loadings are **reconciled**. This means that $Z_S = Z_{1C}$, and $Z_L = Z_{2C}$.When $a_i = 0$ then the quadripole makes not attenuation, and the power at the input is equal to the power at the output. When $a_i < 0$ then the quadripole is considered to be an amplifier.

The main practical solution is the following. If one want to provide the signal transfer through some quadripole with the minimum losses then the input and output impedances have to be balanced, or be equal to the characteristic impedances. The another solution is that the quadripole have to be the amplifier.

5.4.Transmission rate

The rate of output magnitude of some electrical value to the input magnitude of such electrical value is named as the **transmission rate** of some quadripole by given transmission conditions. It is named as **transmission function** or **magnitude-phase characteristic** as well. When the quadripole is amplifier then its transmission rate is usually named as **amplification rate**.

The rate of output and input voltages $K_V=\boldsymbol{V}_2/\boldsymbol{V}_1$ is named also the **voltage transmission rate**. The rate of output and input currents $K_I=\boldsymbol{I}_2/\boldsymbol{I}_1$ is named the **current transmission rate**. If an amplifier is considered then such rates are named as voltage and current amplification rates, respectively.

The transmission rates can be derived from the quadripole coefficients of different forms. The voltage transmission rate for the circuit on the fig. 51 is equal to

$$K_V = \boldsymbol{V}_2/\boldsymbol{V}_1 = Z_L/(A_{11}Z_2+A_{12}) = Z_L/(-Y_{22}Z_L/Y_{21} - 1/Y_{21}) \quad (12)$$

where Z_L is the loading resistance. When $Z_L=\infty$, i.e. the output is open, then

$$K_{V0} = -Y_{21}/Y_{22} = Z_{21}/Z_{11}=1/A_{11}. \quad (13)$$

The current transmission rate is equal to

$$K_I = \boldsymbol{I}_2/\boldsymbol{I}_1 = 1/(A_{21}Z_L+A_{22}) = Z_L/(Z_L/Z_{21}+Z_{22}/Z_{21}). \quad (14)$$

When $Z_L=0$, i.e. the output is shortened, then

$$K_{IS}=Z_{21}/Z_{22}= -Y_{21}/Y_{11}.$$

5.5. Quadripole connections

Cascaded connection of quadripoles is shown on the fig. 52.

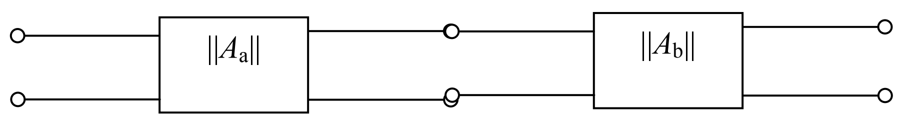


Fig. 52. Cascaded connection of quadripoles

The A – matrix of the resulting quadripole is calculated as

$$||A||=||A_a||||A_b||. \quad (15)$$

When the output characteristic impedance of the first quadripole is equal to the input characteristic impedance of the second quadripole, or $Z_{a2c}=Z_{b1c}$ then the attenuation in decibels of the resulting quadripole are estimated as:

$$a = a_a + a_b\ ;$$

and the transmission rate is equal to:

$$K = K_a K_b\ .$$

Consider a_a=20 db a_b= 12 db then a=20+12 = 32 db.

When $Z_{a2c} \neq Z_{b1c}$ then a higher rate of attenuation occurs.

Sequential connection of quadripoles is shown on the fig. 53.

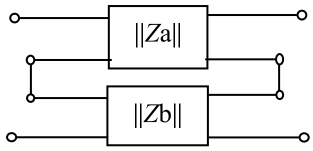


Fig. 53. Sequential connection of quadripoles

The Z – matrix of the resulting quadripole is calculated as

$$||Z||=||Z_a||+||Z_b||. \tag{16}$$

Parallel connection of quadripoles is shown on the fig. 54.

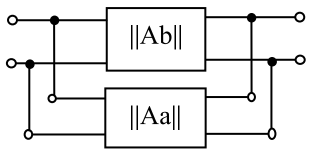


Fig. 54. Parallel connection of quadripoles

The Y – matrix of the resulting quadripole is calculated as

$$||Y||=||Y_a||+||Y_b||. \tag{17}$$

Sequential-parallel connection of quadripoles is illustrated by the fig. 55.

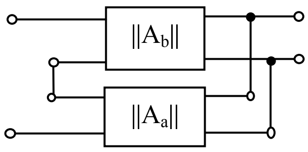


Fig. 56. Sequential-parallel connection of quadripoles

The H – matrix of the resulting quadripole can be derived as

$$||H||=||H_a||+||H_b||. \tag{18}$$

Note, that the shown characteristics of the connected quadripoles are true if and only if the current, which is flowing in, is equal to the current, which is flowing out in both contact couples. This conditions are satisfied if the signal source is conected to the drain circuits only through the considered quadripole, and has not another connections to that drain circuits.

5.6. Simple quadripoles

Simplest quadripoles are shown on the fig. 57.

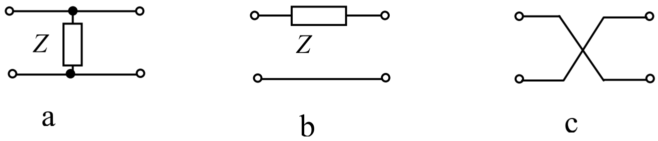


Fig. 57. Simplest quadripoles

The respective *A* - matrices for the quadripoles a,b,c, are the following

$$\|A\| = \begin{pmatrix} 1 & 0 \\ 1/Z & 1 \end{pmatrix}; \qquad \|A\| = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}; \qquad \|A\| = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

Consider the cascaded connection of the quadripoles a and b on the fig 57. It is illustrated by the fig. 58.

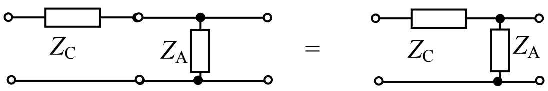


Fig. 58. Cascaded connenction of simplest quadripoles

Then the *A* – matrix of the resulting quadripole will be equal to the product of matrices:

$$\|A\| = \begin{pmatrix} 1 & Z_C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/Z_A & 1 \end{pmatrix} = \begin{pmatrix} 1+Z_C/Z_A & Z_C \\ 1/Z_A & 1 \end{pmatrix}. \tag{20}$$

In such a way one can derive the parameter matrices another simple quadripoles, for example, T-imaged, П-imaged, or bridge quadripoles.

5.7. Feedback Connection

The sequential-parralel quadripole connection of a main quadripole and an auxiliary quadripole represents one of the widely used circuit with the **feedback**. In such a circuit the output voltage $\boldsymbol{V}_2$ is feeded back to the input of the main quadripole through the auxiliary quadripole and inferes the input voltage $\boldsymbol{V}_1$ of the system (see the fig. 59) .

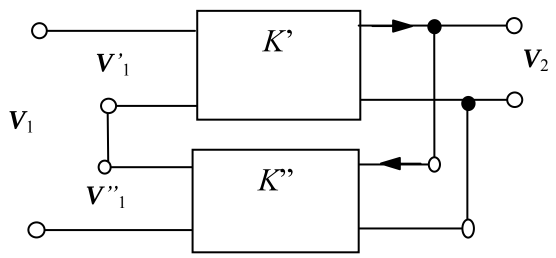


Fig. 59. Sequential-parallel quadripole connection as a feedback circuit

Consider $K' = \boldsymbol{V}_2/\boldsymbol{V}'_1$ is the main device amplification, and $K'' = \boldsymbol{V}''_1/\boldsymbol{V}_2$ is the feedback circuit amplification. The whole circuit input voltage is equal to $\boldsymbol{V}_1 = \boldsymbol{V}'_1 - \boldsymbol{V}''_1$, while the output voltage is the same $\boldsymbol{V}_2$. Then the whole device amplification is

$$K = \boldsymbol{V}_2/\boldsymbol{V}_1 = K'/(1-K'K'') \ . \qquad (21)$$

The **positive feedback** is considered if the output voltage increase forces the input voltage increase, i.e. when $\boldsymbol{V}_1 = \boldsymbol{V}'_1 + K''\boldsymbol{V}_2$, and $K''>0$. The resulting amplification rate increases: $|K|>|K'|$. But in most of cases the positive feedback causes the oscillations in the circuit.

The **negative feedback** is the opposite situation, when $\boldsymbol{V}_1 = \boldsymbol{V}'_1 - K''\boldsymbol{V}_2$. Then the resulting amplification rate decreases, i.e.: $|K|<|K'|$. The equation (20) can be rewritten as

$$K = 1/K'' \cdot (K'K''/(1-K'K'')) \approx -1/K''. \qquad (22)$$

The last ratio in (22) is true if $K'K''>>1$. The equation (22) shows that the amplification rate of the circuit with the feedback can be effectively regulated by the feedback circuit. In such a way the amplification rate of the amplifier can be precisely installed by regulating the feedback transmission rate K''. If the transmission rate of the feedback circuit depends on the frequency then the resulting circuit can serve as the electronic filter.

5.8. AC analysis of quadripoles

In the described above quadripole analysis it was considered that currents can be both DC and AC. The quadripole impedances have both resistive and reactive properties. In such conditions the input current exchange cause some output current exchange. But the output current form not always follows the input current form. Therefore to investigate quadripoles it is useful to put the input current as a time dependent function $\boldsymbol{I}_1(t)$ that is relatively unaffected when passed through any quadripole. The output curve $\boldsymbol{I}_2(t)$ have to be equal to the input curve scaled by some constant *c,* i.e. $\boldsymbol{I}_2(t)=c\,\boldsymbol{I}_1(t)$.

Such a function is called an **eigenfunction,** and the scaling constant *c* is called an **eigenvalue**. It was found out that for electric circuits the following complex function can be used as eigenfunction

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) = \exp(j\omega t).$$

Note that any complex function x can be represented in sine – cosine form as $x = \mathrm{Re}(x)+j\mathrm{Im}(x)$, and in exponential form as $x = |x| \exp(j\varphi)$. Here the **magnitude**

$$|x| = \sqrt{Re^2(x)+Im^2(x)}\ ,$$

and **phase**

$$\varphi = \mathrm{arctg}(\mathrm{Im}(x)/\mathrm{Re}(x)).$$

The current transmission rate of the quadripole can be described at the frequency ω as $K_\mathrm{I}(\omega) = k \exp(j\varphi)$. Then the output current is equal to

$$\boldsymbol{I}_2(j\omega t) = k \exp(j\varphi) \exp()\, \boldsymbol{I}_1(j\omega t) = k \exp(j(\omega t+\varphi))\, \boldsymbol{I}_1(j\omega t).$$

Here k is the eigenvalue of the transmission rate K_I at the frequency ω. All such eigenvalues at a set of frequencies form the **magnitude-frequency characteristic** of the quadripole, or in other words, the **transmission spectrum function**. Respectively, all the phase φ values form the **phase-frequency characteristic,** or **phase spectrum function.**

Similar dependences can be derived for the voltage transmission rate and amplification rate of the quadripole. But they will be different on each other in the phase spectrum function.

For example, consider the quadripole on the fig. 58, in which $Z_\mathrm{A}= j\,\omega L$, and $Z_\mathrm{C} =1/(j\,\omega C)$. Then the voltage transmission rate of the quadripole, loaded by Z_L due to (12) and (20) is

$$K_\mathrm{V}(\omega) = Z_\mathrm{L}/(A_{11}Z_\mathrm{L}+A_{12}) = Z_\mathrm{L}/((1+ Z_\mathrm{A}/Z_\mathrm{C})\, Z_\mathrm{L} + Z_\mathrm{C}) = Z_\mathrm{L}/((1 - \omega^2 LC)\, Z_\mathrm{L} - j/(\omega C)).$$

Consider Z_L=500 ohm, C=100 microfarads, L= 1 Henry. The derived magnitude-frequency characteristic is shown on the fig. 60.

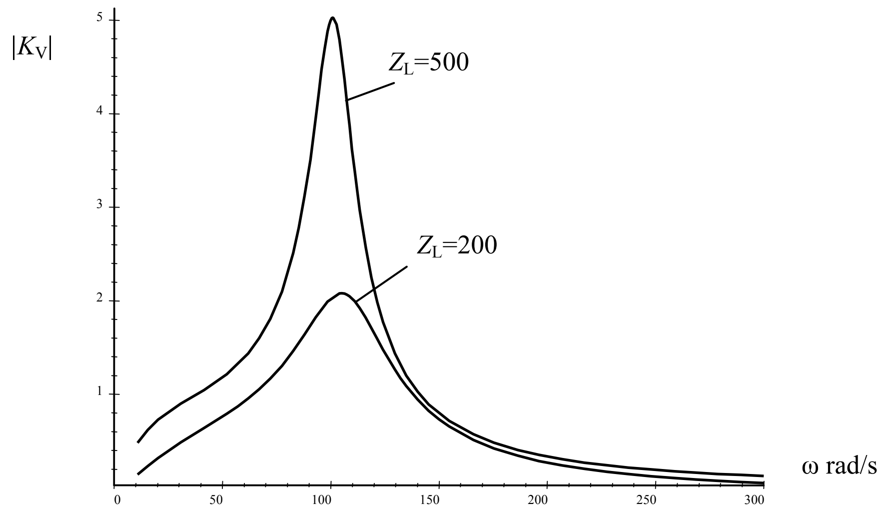


Fig. 60. Magnitude-frequency characteristic of the quadripole on the fig. 58

The fig 60 shows the frequency selection properties of derived quadripole. This quadripole transmits, and even amplifies the signal with the central frequency of 100 rad/s, and suppress another frequencies, which are far from the central frequency. One can see that when the loading resistance is decreased to 200 ohm then the selectivity properties drops substantially. This example shows the important role of the quadripole loading.

In such a way any quadripole can be analyzed. And the calculation of the analytical form of the spectrum function helps to synthesize the quadripole with the given spectrum properties.